Appendice 9

CALCOLO di $\int_{-1}^{1} P_j^{m-q}(x)\cdot P_j^{m+q}(x)dx$

Sviluppo del binomio $(x^2-1)^j$ per ottenere un polinomio pari di j+1 termini ordinati secondo un indice i ($0\leq i\leq j$) crescente associato a potenze decrescenti di x.

Il binomio di Newton : $(y-1)^j = \sum_{h=0}^{j} (-1)^{j-h}\binom{j}{h} y^h$ con $y = x^2$ porta a

$(x^2-1)^j = \sum_{h=0}^{j} (-1)^{j-h}\binom{j}{h} x^{2h}$ oppure a per i=j-h $(x^2-1)^j = \sum_{i=0}^{j} (-1)^{i}\binom{j}{i} x^{2j-2i}$

$$\frac{d}{dx}(x^2-1)^j = \sum_{i=0}^{j} (-1)^{i}\binom{j}{i}(2j-2i)x^{2j-2i-1} \quad \text{per } 2j-2i-1 > 0 \quad i < j - \frac{1}{2}$$

$$\frac{d}{dx}(x^2-1)^j = \sum_{i=0}^{\mathrm{int}(j-\frac{1}{2})} (-1)^{i}\binom{j}{i}(2j-2i)x^{2j-2i-1}$$

dove int $[j-\frac{1}{2}]$ indica la parte intera dell'argomento $j-\frac{1}{2}$.

$$\frac{d^2}{dx^2}[(x^2-1)^j] = \sum_{i=1}^{\mathrm{int}(j-\frac{2}{2})} (-1)^{i}\binom{j}{i}\cdot 2i\cdot(2i-1)\cdot x^{2j-2i-2} \quad \text{per } 2j-2i-2 \geq 0$$

$$\frac{d^3}{dx^3}[(x^2-1)^j] = \sum_{i=1}^{\mathrm{int}(j-\frac{3}{2})} (-1)^{i}\binom{j}{i}\cdot 2i\cdot(2i-1)\cdot(2i-2)\cdot x^{2j-2i-3} \quad \text{per } 2j-2i-3 \geq 0$$

La derivata di ordine s ha la forma:

$$\frac{d^s}{dx^s}[(x^2-1)^j] = \sum_{i=0}^{\mathrm{int}[j-\frac{s}{2}]} (-1)^{i}\binom{j}{i}(2j-2i)(2j-2i-1)\cdots(2j-2i-s+1)\cdot x^{2j-2i-s}$$

Per s = j la precedente formula definisce il polinomio di Legendre $P_j(x)$ scrivendo:

$$P_j(x) = \frac{1}{2^j\cdot j!}\cdot\frac{d^j}{dx^j}[(x^2-1)^j] = \frac{1}{2^j\cdot j!}\cdot\sum_{i=0}^{int[\frac{j}{2}]}(-1)^i\binom{j}{i}(2j-2i)(2j-2i-1)\cdots(j-2i+1)\cdot x^{j-2i}$$

Un'ulteriore derivazione eseguita m volte definisce la funzione di Legendre associata $P_j^m(x)$ mediante la formula:

$$2^j\cdot j!\cdot(1-x^2)^{-\frac{m}{2}}\cdot P_j^m(x) = \sum_{i=0}^{int[\frac{j-m}{2}]}(-1)^i\binom{j}{i}(2j-2i)(2j-2i-1)\cdots(j-2i-m+1)\cdot x^{j-2i-m}$$

Il valore di m può essere sostituito da m-q oppure da m+q ottenendo rispettivamente:

$$2^j\cdot j!\cdot(1-x^2)^{-\frac{m-q}{2}}\cdot P_j^{m-q}(x) = \sum_{i=0}^{int[\frac{j-m+q}{2}]}(-1)^i\binom{j}{i}(2j-2i)(2j-2i-1)\cdots(j-2i-m+q+1)\cdot x^{j-2i-m+q}$$

$$2^j\cdot j!\cdot(1-x^2)^{-\frac{m+q}{2}}\cdot P_j^{m+q}(x) = \sum_{h=0}^{int[\frac{j-m-q}{2}]}(-1)^h\binom{j}{h}(2j-2h)(2j-2h-1)\cdots(j-2h-m-q+1)\cdot x^{j-2h-m-q}$$

Si possono ora indicare i coefficienti di ciascun polinomio in modo semplificato ponendo:

$$(-1)^i\binom{j}{i}(2j-2i)(2j-2i-1)\cdots(j-2i-m+q+1) = G_{j,m,-q,i}$$

$$(-1)^h\binom{j}{h}(2j-2h)(2j-2h-1)\cdots(j-2h-m-q+1) = G_{j,m,q,h}$$

per ottenere:

$$2^j\cdot j!\cdot(1-x^2)^{-\frac{m-q}{2}}\cdot P_j^{m-q}(x) = \sum_{i=0}^{int[\frac{j-m+q}{2}]} G_{j,m,-q,i}\cdot x^{j-2i-m+q}$$

$$2^j\cdot j!\cdot(1-x^2)^{-\frac{m+q}{2}}\cdot P_j^{m+q}(x) = \sum_{h=0}^{int[\frac{j-m-q}{2}]} G_{j,m,q,h}\cdot x^{j-2h-m-q}$$

Il prodotto membro a membro delle ultime due equazioni porta a:

$$[2^j\cdot j!]^2\cdot(1-x^2)^{-m}P_j^{m-q}(x)\cdot P_j^{m+q}(x) =$$

$$= \sum_{i=0}^{\mathrm{int}\left[\frac{j-m+q}{2}\right]} G_{j,m,-q,i}\cdot x^{j-2i-m+q}\cdot \sum_{h=0}^{\mathrm{int}\left[\frac{j-m-q}{2}\right]} G_{j,m,q,h}\cdot x^{j-2h-m-q}$$

che si può anche scrivere come segue:

$$P_j^{m-q}(x)\cdot P_j^{m+q}(x) = \left[\frac{1}{2^j\cdot j!}\right]^2\cdot \sum_{i=0}^{\mathrm{int}\left[\frac{j-m+q}{2}\right]} \sum_{h=0}^{\mathrm{int}\left[\frac{j-m-q}{2}\right]} G_{j,m,-q,i}G_{j,m,q,h}\cdot x^{2(j-m-i-h)}(1-x^2)^m$$

Per calcolare l'integrale $\int_{-1}^{1} P_j^{m-q}(x)\cdot P_j^{m+q}(x)dx$ conviene calcolare separatamente quello dei fattori dipendenti da x contenuti nelle precedenti sommatorie e porre:

$$\int_{-1}^{1} S_{i,h}^{j,m}(x)dx = \int_{-1}^{1} x^{2(j-m-i-h)}(1-x^2)^m dx = 2\int_0^1 x^{2(j-m-i-h)-1}(1-x^2)^m x dx = \qquad ;\ x^2 = y$$

$$= \int_0^1 y^{j-m-i-h-\frac{1}{2}}(1-y)^m dy = \int_0^1 y^{a-1}(1-y)^{b-1}dy = \beta(a,b) = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)} \quad ;\ j-m-i-h+\frac{1}{2} = a \quad ;\ m+1 = b$$

$$\int_{-1}^{1} S_{i,h}^{j,m}(x)dx = \beta(a,b) = \frac{\Gamma(j-m-i-h+\frac{1}{2})\cdot\Gamma(m+1)}{\Gamma(j-i-h+\frac{3}{2})} = \frac{\Gamma(j-m-i-h+\frac{1}{2})\cdot m!}{\Gamma(j-i-h+\frac{3}{2})} \quad ;$$

Mediante una deduzione dalla formula dei complementi si può far scomparire la funzione Γ dalla doppia sommatoria:

$$\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{2^{2n}\cdot n!}\cdot\sqrt{\pi}$$

$$\Gamma(j-m-i-h+\frac{1}{2}) = \frac{[2(j-m-i-h)]!}{2^{2(j-m-i-h)}\cdot(j-m-i-h)!}\cdot\sqrt{\pi}$$

$$\Gamma(j-i-h+\frac{3}{2}) = \Gamma(j-i-h+1+\frac{1}{2}) = \frac{[2(j-i-h+1)]!}{2^{2(j-i-h+1)}\cdot(j-i-h+1)!}\cdot\sqrt{\pi}$$

$$\frac{\Gamma(j-m-i-h+\frac{1}{2})}{\Gamma(j-i-h+\frac{3}{2})} = 2^{2(m+1)}\cdot\frac{[2(j-m-i-h)]!\cdot(j-i-h+1)!}{[2(j-i-h+1)]!\cdot(j-m-i-h)!}$$

posto $j-i-h = s$

$$2^{2(m+1)}\cdot\frac{[2(j-m-i-h)]!\cdot(j-i-h+1)!}{[2(j-i-h+1)]!\cdot(j-m-i-h)!} = 2^{2(m+1)}\cdot\frac{[2(s-m)]!(s+1)!}{[2(s+1)]!(s-m)!} =$$

$$2^{2(m+1)}\cdot\frac{2^{s-m}[2(s-m)-1][2(s-m)-3][2(s-m)-5]\cdots}{2^{s+1}[2(s+1)-1][2(s+1)-3][2(s+1)-5]\cdots} =$$

$$= 2^{m+1}\cdot\frac{[2(s-m)-1][2(s-m)-3][2(s-m)-5]\cdots}{(2s+1)(2s-1)(2s-3)\cdots} = \frac{2^{m+1}}{(2s+1)(2s-1)(2s-3)\cdots(2s-2m+1)} =$$

$$= \frac{1}{(s+\frac{1}{2})(s-\frac{1}{2})(s-\frac{3}{2})\cdots(s-m+\frac{1}{2})} = \frac{1}{(s+\frac{1}{2})[(s+\frac{1}{2})-1][(s+\frac{1}{2})-2]\cdots[(s+\frac{1}{2})-m]}$$

$$\int_{-1}^{1} S^{j,m}_{i,h}(x)dx = \frac{\Gamma(j-m-i-h+\frac{1}{2})\cdot m!}{\Gamma(j-i-h+\frac{3}{2})} = \frac{m!}{(j-i-h+\frac{1}{2})(j-i-h+\frac{1}{2}-1)\cdots(j-i-h+\frac{1}{2}-m)}$$

Si può ora passare al calcolo di:

$$\int_{-1}^{1} P^{m-q}_{j}(x)\cdot P^{m+q}_{j}(x)dx = \left[\frac{1}{2^{j}\cdot j!}\right]^{2}\cdot\sum_{i=0}^{int[\frac{j-m+q}{2}]}\ \sum_{h=0}^{int[\frac{j-m-q}{2}]} G_{j,m,-q,i}G_{j,m,q,h}\cdot\int_{-1}^{1} S^{j,m}_{i,h}(x)dx$$

Visto che l'integrale a primo membro è riducibile ad una somma di somme finite si può indicare con il simbolo dotato degli indici non saturati j,m,q scrivendo:

$$\sum\begin{pmatrix} j\\ m\\ q\end{pmatrix} = \int_{-1}^{1} P^{m-q}_{j}(x)\cdot P^{m+q}_{j}(x)dx$$

Si può ora valutare il prodotto: $G_{j,m,-q,i}\cdot G_{j,m,q,h}\cdot\int_{-1}^{1} S^{j,m}_{i,h}(x)dx$

$$G_{j,m,-q,i}\cdot G_{j,m,q,h} = (-1)^{i+h}\cdot\frac{(j!)^{2}}{i!(j-i)!\cdot h!(j-h)!}\cdot\frac{(2j-2i)!}{(j-2i-m+q)!}\cdot\frac{(2j-2h)!}{(j-2h-m-q)!} =$$

$$= (-1)^{i+h}\cdot\binom{j}{i}\cdot\binom{j}{h}\cdot(j+m-q)!\cdot\binom{2j-2i}{j+m-q}\cdot(j+m+q)!\cdot\binom{2j-2h}{j+m+q}$$

$$G_{j,m,-q,i}\cdot G_{j,m,q,h}\cdot\int_{-1}^{1} S^{j,m}_{i,h}(x)dx =$$

$$= (j+m-q)!\cdot(j+m+q)!\cdot(-1)^{i+h}\cdot\binom{j}{i}\cdot\binom{j}{h}\cdot\binom{2j-2i}{j+m-q}\cdot\binom{2j-2h}{j+m+q}\cdot\frac{\Gamma(j-m-i-h+\frac{1}{2})\cdot m!}{\Gamma(j-i-h+\frac{3}{2})} =$$

$$= (j+m-q)!\cdot(j+m+q)!\cdot m!\cdot\frac{(-1)^{i+h}\cdot\binom{j}{i}\cdot\binom{j}{h}\cdot\binom{2j-2i}{j+m-q}\cdot\binom{2j-2h}{j+m+q}}{(j-i-h+\frac{1}{2})(j-i-h+\frac{1}{2}-1)\cdots(j-i-h+\frac{1}{2}-m)}$$

I fattori (j+m-q)!·(j+m+q)!·m! sono indipendenti dagli indici i ed h delle sommatorie e ne possono essere estratti. Da ciò consegue:

$$\Sigma\begin{pmatrix} j \\ m \\ q \end{pmatrix} = \frac{(j+m-q)!\cdot(j+m+q)!\cdot m!}{2^{2j}\cdot(j!)^2}\cdot\sum_{i=0}^{\mathrm{int}\left[\frac{j-m+q}{2}\right]}\sum_{h=0}^{\mathrm{int}\left[\frac{j-m-q}{2}\right]}\frac{(-1)^{i+h}\cdot\binom{j}{i}\cdot\binom{j}{h}\cdot\binom{2j-2i}{j+m-q}\cdot\binom{2j-2h}{j+m+q}}{(j-i-h+\frac{1}{2})(j-i-h+\frac{1}{2}-1)\cdots(j-i-h+\frac{1}{2}-m)}$$