Appendice 7

Definizione mediante formula di Rodriguez dei polinomi di Laguerre $L_i^{\alpha}(\rho)$ ortogonali sull'intervallo $0 < \rho < \infty$ con peso nell'integrale del prodotto scalare $e^{-\rho}\cdot\rho^{\alpha}$:

$$L_i^{\alpha}(\rho) = \frac{1}{i!}\cdot e^{\rho}\cdot\rho^{-\alpha}\cdot\frac{d^i}{d\rho^i}\left(e^{-\rho}\cdot\rho^{i+\alpha}\right)$$

Ortogonalità

$$\int_0^{\infty} e^{-\rho}\cdot\rho^{\alpha}\cdot L_j^{\alpha}(\rho)\cdot L_k^{\alpha}(\rho)\,d\rho = \int_0^{\infty} e^{-\rho}\cdot\rho^{\alpha}\cdot\frac{1}{j!}\cdot e^{\rho}\cdot\rho^{-\alpha}\cdot\frac{d^j}{d\rho^j}\left(e^{-\rho}\cdot\rho^{j+\alpha}\right) L_k^{\alpha}(\rho)\,d\rho =$$

$$= \frac{1}{j!}\cdot\int_0^{\infty}\frac{d^j}{d\rho^j}\left(e^{-\rho}\cdot\rho^{j+\alpha}\right) L_k^{\alpha}(\rho)\,d\rho = \frac{1}{j!}\cdot\frac{d^{j-1}}{d\rho^{j-1}}\left(e^{-\rho}\cdot\rho^{j+\alpha}\right)\cdot L_k^{\alpha}(\rho)\Big|_{\rho=0}^{\rho=\infty} +$$

$$- \frac{1}{j!}\cdot\int_0^{\infty}\frac{d^{j-1}}{d\rho^{j-1}}\left(e^{-\rho}\cdot\rho^{j+\alpha}\right)\cdot\frac{d}{d\rho}\left[L_k^{\alpha}(\rho)\right]d\rho = \frac{1}{j!}\cdot\int_0^{\infty}\frac{d^{j-1}}{d\rho^{j-1}}\left(e^{-\rho}\cdot\rho^{j+\alpha}\right)\cdot L_{k-1}^{\alpha+1}(\rho)\,d\rho$$

Integrazione per parti ricorsiva:

$$\frac{1}{j!}\cdot\int_0^{\infty}\frac{d^{j-1}}{d\rho^{j-1}}\left(e^{-\rho}\cdot\rho^{j+\alpha}\right)\cdot L_{k-1}^{\alpha+1}(\rho)\,d\rho = \frac{1}{j!}\cdot\int_0^{\infty}\frac{d^{j-h}}{d\rho^{j-h}}\left(e^{-\rho}\cdot\rho^{j+\alpha}\right)\cdot L_{k-h}^{\alpha+h}(\rho)\,d\rho =$$

$$= \frac{1}{j!}\cdot\int_0^{\infty}\frac{d^{j-j}}{d\rho^{j-j}}\left(e^{-\rho}\cdot\rho^{j+\alpha}\right)\cdot L_{k-j}^{\alpha+j}(\rho)\,d\rho = \frac{1}{j!}\cdot\int_0^{\infty} e^{-\rho}\cdot\rho^{j+\alpha}\cdot L_{k-j}^{\alpha+j}(\rho)\,d\rho$$

$$\int_0^{\infty} e^{-\rho}\cdot\rho^{\alpha}\cdot L_j^{\alpha}(\rho)\cdot L_k^{\alpha}(\rho)\,d\rho = \frac{1}{j!}\cdot\int_0^{\infty} e^{-\rho}\cdot\rho^{j+\alpha}\cdot L_{k-j}^{\alpha+j}(\rho)\,d\rho$$

$L_{k-j}^{\alpha+j}(\rho) = 0$ per $j>k$ perché $L_0^{\alpha+j}(\rho) = 1$

analogamente deve valere la relazione:

$$\int_0^{\infty} e^{-\rho}\cdot\rho^{\alpha}\cdot L_j^{\alpha}(\rho)\cdot L_k^{\alpha}(\rho)\,d\rho = \frac{1}{k!}\cdot\int_0^{\infty} e^{-\rho}\cdot\rho^{k+\alpha}\cdot L_{j-k}^{\alpha+k}(\rho)\,d\rho$$

$L_{j-k}^{\alpha+k}(\rho) = 0$ per $k>j$ perché $L_0^{\alpha+k}(\rho) = 1$

L'unico caso per cui può essere $\int_0^\infty e^{-\rho}\cdot\rho^\alpha\cdot L_j^\alpha(\rho)\cdot L_k^\alpha(\rho)\,d\rho \neq 0$ si ha per j = k ed in tal caso:

$$\int_0^\infty e^{-\rho}\cdot\rho^\alpha\cdot\left[L_j^\alpha(\rho)\right]^2 d\rho = \frac{1}{j!}\cdot\int_0^\infty e^{-\rho}\cdot\rho^{j+\alpha}\,d\rho = \frac{\Gamma(\alpha+j+1)}{j!}$$

Calcolo dei primi tre polinomi

$$L_0^\alpha(\rho) = \frac{1}{0!}\cdot e^{\rho}\cdot\rho^{-\alpha}\cdot(e^{-\rho}\cdot\rho^{\alpha}) = 1$$

$$L_1^\alpha(\rho) = \frac{1}{1!}\cdot e^{\rho}\cdot\rho^{-\alpha}\cdot\frac{d}{d\rho}(e^{-\rho}\cdot\rho^{1+\alpha}) = e^{\rho}\cdot\rho^{-\alpha}\cdot[(1+\alpha)\cdot e^{-\rho}\cdot\rho^{\alpha} - e^{-\rho}\cdot\rho^{1+\alpha}] = 1+\alpha-\rho$$

$$L_2^\alpha(\rho) = \frac{1}{2!}\cdot e^{\rho}\cdot\rho^{-\alpha}\cdot\frac{d^2}{d\rho^2}(e^{-\rho}\cdot\rho^{2+\alpha}) = \frac{1}{2!}\cdot e^{\rho}\cdot\rho^{-\alpha}\cdot\frac{d}{d\rho}[e^{-\rho}\cdot\rho^{1+\alpha}(2+\alpha-\rho)] =$$

$$= \frac{1}{2}\cdot e^{\rho}\cdot\rho^{-\alpha}\cdot[e^{-\rho}\cdot(1+\alpha)\rho^{\alpha}(2+\alpha-\rho) - e^{-\rho}\cdot\rho^{1+\alpha} - e^{-\rho}\cdot\rho^{1+\alpha}(2+\alpha-\rho)] =$$

$$= \frac{1}{2}\cdot[(1+\alpha)(2+\alpha-\rho) - \rho - \rho(2+\alpha-\rho)] = \frac{1}{2}\cdot[(1+\alpha)(2+\alpha) - \rho(4+2\alpha) + \rho^2]$$

$$L_3^\alpha(\rho) = \frac{1}{3!}\cdot e^{\rho}\cdot\rho^{-\alpha}\cdot\frac{d^3}{d\rho^3}(e^{-\rho}\cdot\rho^{3+\alpha}) = \frac{1}{6}\cdot e^{\rho}\cdot\rho^{-\alpha}\cdot\frac{d^2}{d\rho^2}\{e^{-\rho}[(3+\alpha)\rho^{2+\alpha} - \rho^{3+\alpha}]\} =$$

$$= \frac{1}{6}\cdot e^{\rho}\cdot\rho^{-\alpha}\cdot\frac{d}{d\rho}\{e^{-\rho}[(3+\alpha)(2+\alpha)\rho^{1+\alpha} - 2(3+\alpha)\rho^{2+\alpha} + \rho^{3+\alpha}]\} =$$

$$= \frac{1}{6}\cdot e^{\rho}\cdot\rho^{-\alpha}\cdot\{e^{-\rho}[(3+\alpha)(2+\alpha)(1+\alpha)\rho^{\alpha} - 3(3+\alpha)(2+\alpha)\rho^{1+\alpha} + 3(3+\alpha)\rho^{2+\alpha} - \rho^{3+\alpha}]\} =$$

$$= \frac{1}{6}\cdot[(3+\alpha)(2+\alpha)(1+\alpha) - 3(3+\alpha)(2+\alpha)\rho + 3(3+\alpha)\rho^2 - \rho^3]$$

Primi 5 poliomi di Laguerre con i coefficeiti numerici di $(a-b)^i$ per il polinomio di ordine i (e continuazione per induzione)

$$L_0^\alpha(\rho) = 1$$

$$L_1^\alpha(\rho) = 1+\alpha-\rho$$

$$L_2^\alpha(\rho) = \frac{1}{2}\cdot[(2+\alpha)(1+\alpha) - 2(2+\alpha)\rho + \rho^2]$$

$$L_3^\alpha(\rho) = \frac{1}{6}\cdot[(3+\alpha)(2+\alpha)(1+\alpha) - 3(3+\alpha)(2+\alpha)\rho + 3(3+\alpha)\rho^2 - \rho^3]$$

$$L_4^\alpha(\rho) = \frac{1}{24}\cdot[(4+\alpha)(3+\alpha)(2+\alpha)(1+\alpha) - 4(4+\alpha)(3+\alpha)(2+\alpha)\rho + 6(4+\alpha)(3+\alpha)\rho^2 +$$
$$- 4(4+\alpha)\rho^3 + \rho^4]$$

$$L_5^\alpha(\rho) = \frac{1}{120}\cdot[(5+\alpha)(4+\alpha)(3+\alpha)(2+\alpha)(1+\alpha) - 5(5+\alpha)(4+\alpha)(3+\alpha)(2+\alpha)\rho +$$
$$+ 10(5+\alpha)(4+\alpha)(3+\alpha)\rho^2 - 10(5+\alpha)(4+\alpha)\rho^3 + 5(5+\alpha)\rho^4 - \rho^5]$$

$$L_5^{\alpha}(\rho) = (-1)^5 \frac{1}{120}\cdot\rho^5 + (-1)^4 \frac{5(5+\alpha)}{120}\rho^4 + \cdot(-1)^3\frac{10}{120}(5+\alpha)(4+\alpha)\rho^3 + \cdots\cdots =$$

$$= \frac{1}{5!}\left[(-1)^5\binom{5}{5}\cdot\rho^5 + (-1)^4\cdot\binom{5}{4}\cdot\frac{(5+\alpha)!}{(4+\alpha)!}\rho^4 + (-1)^3\cdot\binom{5}{3}\cdot\frac{(5+\alpha)!}{(3+\alpha)!}\rho^3 + (-1)^2\cdot\binom{5}{2}\cdot\frac{(5+\alpha)!}{(2+\alpha)!}\rho^2 + \right.$$

$$\left. + (-1)^1\cdot\binom{5}{1}\cdot\frac{(5+\alpha)!}{(1+\alpha)!}\rho + (-1)^0\cdot\binom{5}{0}\cdot\frac{(5+\alpha)!}{(0+\alpha)!}\right] = \frac{1}{5!}\cdot\sum_{j=0}^{5}(-1)^j\cdot\binom{5}{j}\cdot\frac{(\alpha+5)!}{(\alpha+j)!}\cdot\rho^j$$

$$L_i^{\alpha}(\rho) = \frac{1}{i!}\cdot\sum_{j=0}^{i}(-1)^j\cdot\binom{i}{j}\cdot\frac{(\alpha+i)!}{(\alpha+j)!}\cdot\rho^j \;;$$

$$L_i^{\alpha}(\rho) = \sum_{j=0}^{i}\frac{(\alpha+i)!}{(i-j)!(\alpha+j)!}\cdot\frac{(-\rho)^j}{j!} = \sum_{j=0}^{i}\frac{(i+\alpha)!}{(j+\alpha)!(i-j)!}\cdot\frac{(-\rho)^j}{j!}$$

$$L_i^{\alpha}(\rho) = \sum_{j=0}^{i}(-1)^j\binom{i+\alpha}{i-j}\cdot\frac{\rho^j}{j!} \quad ; \quad L_i^{\alpha}(\rho) = \sum_{j=0}^{i}\binom{i+\alpha}{i-j}\cdot\frac{(-\rho)^j}{j!}$$

(α non intero) $$L_i^{\alpha}(\rho) = \sum_{j=0}^{i}\frac{\Gamma(i+\alpha+1)}{\Gamma(j+\alpha+1)\cdot(i-j)!}\cdot\frac{(-\rho)^j}{j!}$$

Normalizzazione

$$\int_0^{\infty}\left[L_i^{\alpha}(\rho)\right]^2 e^{-\rho}\cdot\rho^{\alpha}d\rho = \int_0^{\infty}L_i^{\alpha}(\rho)\left[\frac{(-1)^i}{i!}\cdot\rho^i + (-1)^{i-1}\frac{i+\alpha}{(i-1)!}\rho^{i-1} + \cdots\cdots\right]\cdot e^{-\rho}\rho^{\alpha}\,d\rho =$$

$$= \frac{(-1)^i}{i!}\int_0^{\infty}L_i^{\alpha}(\rho)\,\rho^i\cdot e^{-\rho}\,\rho^{\alpha}\,d\rho = \frac{1}{i!}\cdot\frac{(-1)^i}{i!}\int_0^{\infty}e^{\rho}\cdot\frac{d^i}{d\rho^i}\left(e^{-\rho}\cdot\rho^{i+\alpha}\right)\cdot e^{-\rho}\rho^i\,d\rho =$$

$$= \frac{1}{i!}\cdot\frac{(-1)^i}{i!}\int_0^\infty \frac{d^i}{d\rho^i}(e^{-\rho}\cdot\rho^{i+\alpha})\cdot\rho^i\,d\rho =$$

$$= \frac{1}{i!}\cdot\frac{(-1)^i}{i!}\cdot\frac{d^{i-1}}{d\rho^{i-1}}(e^{-\rho}\cdot\rho^{i+\alpha})\,\rho^i\Big|_0^\infty - \frac{1}{i!}\cdot\frac{(-1)^i}{i!}\cdot i\int_0^\infty \frac{d^{i-1}}{d\rho^{i-1}}(e^{-\rho}\cdot\rho^{i+\alpha})\,\rho^{i-1}\,d\rho =$$

$$= (-1)\cdot\frac{1}{i!}\cdot\frac{(-1)^i}{i!}\cdot i\int_0^\infty \frac{d^{i-1}}{d\rho^{i-1}}(e^{-\rho}\cdot\rho^{i+\alpha})\,\rho^{i-1}\,d\rho =$$

$$= (-1)^i\cdot\frac{1}{i!}\cdot\frac{(-1)^i}{i!}\cdot i!\int_0^\infty e^{-\rho}\cdot\rho^{i+\alpha}\,d\rho = \frac{\Gamma(i+\alpha+1)}{i!}$$

$$\frac{i!}{\Gamma(\alpha+i+1)}\int_0^\infty \left[L_i^\alpha(\rho)\right]^2 e^{-\rho}\cdot\rho^\alpha\,d\rho = 1$$

Derivazione

$$L_i^\alpha(\rho) = \sum_{j=0}^{i}\binom{i+\alpha}{i-j}\frac{(-\rho)^j}{j!} = \binom{i+\alpha}{i} + \sum_{j=1}^{i}\binom{i+\alpha}{i-j}\frac{(-\rho)^j}{j!}$$

$$\frac{d}{d\rho}[L_i^\alpha(\rho)] = -\sum_{j=1}^{i}\binom{i+\alpha}{i-j}\frac{(-\rho)^{j-1}}{(j-1)!} \qquad j=q+1$$

$$\frac{d}{d\rho}[L_i^\alpha(\rho)] = -\sum_{q+1=1}^{i}\binom{i+\alpha}{i-q-1}\frac{(-\rho)^{q+1-1}}{(q+1-1)!} = -\sum_{q=0}^{i-1}\binom{i+\alpha}{i-q-1}\frac{(-\rho)^q}{q!} = -\sum_{q=0}^{i-1}\binom{i-1+\alpha+1}{i-1-q}\frac{(-\rho)^q}{q!}$$

$$\frac{d}{d\rho}[L_i^\alpha(\rho)] = -L_{i-1}^{\alpha+1}(\rho)$$

$$\frac{d^p}{d\rho^p}[L_j^\alpha(\rho)] = (-1)^p\cdot L_{j-p}^{\alpha+p}(\rho) \qquad j\geq p$$

Come si è già visto per i polinomi ortogonali classici di Legendre, anche i polinomi di Laguerre producono per derivazione nuovi polinomi ortogonali sullo stesso intervallo d'integrazione, poiché le loro derivate sono ancora polinomi di Laguerre con indici mutati.

Equazione differenziale per i polinomi di Laguerre

$$L_i^{\alpha}(\rho) = \sum_{j=0}^{i} (-1)^j \binom{i+\alpha}{i-j} \frac{\rho^j}{j!}$$

$$\frac{d}{d\rho}[L_i^{\alpha}(\rho)] = -L_{i-1}^{\alpha+1}(\rho) = -\sum_{j=0}^{i-1} (-1)^j \binom{i+\alpha}{i-1-j} \frac{\rho^j}{j!}$$

$$\frac{d^2}{d\rho^2}[L_i^{\alpha}(\rho)] = L_{i-2}^{\alpha+2}(\rho) = \sum_{j=0}^{i-2} (-1)^j \binom{i+\alpha}{i-2-j} \frac{\rho^j}{j!}$$

$$A\frac{d^2}{d\rho^2}[L_i^{\alpha}(\rho)] + B\frac{d}{d\rho}[L_i^{\alpha}(\rho)] + C\,L_i^{\alpha}(\rho) =$$

$$= A\sum_{j=0}^{i-2} (-1)^j \binom{i+\alpha}{i-2-j} \frac{\rho^j}{j!} - B\sum_{j=0}^{i-1} (-1)^j \binom{i+\alpha}{i-1-j} \frac{\rho^j}{j!} + C\sum_{j=0}^{i} (-1)^j \binom{i+\alpha}{i-j} \frac{\rho^j}{j!} =$$

$$= -B\cdot(-1)^{i-1}\binom{i+\alpha}{0}\frac{\rho^{i-1}}{(i-1)!} + C(-1)^i\binom{i+\alpha}{0}\frac{\rho^i}{i!} + C\cdot(-1)^{i-1}\binom{i+\alpha}{1}\frac{\rho^{i-1}}{(i-1)!} +$$

$$+ \sum_{j=0}^{i-2} (-1)^j \left[A\binom{i+\alpha}{i-2-j} - B\binom{i+\alpha}{i-1-j} + C\binom{i+\alpha}{i-j}\right]\cdot\frac{\rho^j}{j!} =$$

$$=(-1)^i\cdot\left[B\cdot\frac{\rho^{i-1}}{(i-1)!} + C\cdot\frac{\rho^i}{i!} - C\cdot(i+\alpha)\cdot\frac{\rho^{i-1}}{(i-1)!}\right] + \sum_{j=0}^{i-2} (-1)^j \left[A\binom{i+\alpha}{i-2-j} - B\binom{i+\alpha}{i-1-j} + C\binom{i+\alpha}{i-j}\right]\cdot\frac{\rho^j}{j!}$$

$$\left[B\cdot\frac{\rho^{i-1}}{(i-1)!} + C\cdot\frac{\rho^i}{i!} - C\cdot(i+\alpha)\cdot\frac{\rho^{i-1}}{(i-1)!}\right] = \frac{\rho^{i-1}}{(i-1)!}\left[B + C\cdot\frac{\rho}{i} - C\cdot(i+\alpha)\right] = \frac{\rho^{i-1}}{(i-1)!}\left[B + \rho - i(i+\alpha)\right]$$

uno dei coefficienti si può fissare arbitrariamente e indifferentemente rispetto alla nullità del termine noto. Per C=i si ha:

$$A\frac{d^2}{d\rho^2}[L_i^\alpha(\rho)] + B\frac{d}{d\rho}[L_i^\alpha(\rho)] + i\,L_i^\alpha(\rho) =$$

$$= (-1)^i\cdot[B+\rho - i(i+\alpha)]\cdot\frac{\rho^{i-1}}{(i-1)!} + \sum_{j=0}^{i-2}(-1)^j\left[A\binom{i+\alpha}{i-2-j} - B\binom{i+\alpha}{i-1-j} + i\binom{i+\alpha}{i-j}\right]\cdot\frac{\rho^j}{j!}$$

per $B+\rho = N$ indipendente da ρ si annulla il termine in ρ^i

$$\binom{i+\alpha}{i-j-1} + \binom{i+\alpha}{i-j-2} = \frac{(i+\alpha)!\,(j+\alpha+2)}{(i-j-1)!(j+\alpha+2)!} + \frac{(i-j-1)(i+\alpha)!}{(i-j-1)!(j+\alpha+2)!} = \frac{(i+\alpha+1)!}{(i-j-1)!(j+\alpha+2)!} = \binom{i+\alpha+1}{i-j-1}$$

$$\binom{i+\alpha}{i-j-2} = \binom{i+\alpha+1}{i-j-1} - \binom{i+\alpha}{i-j-1}$$

$$A\frac{d^2}{d\rho^2}[L_i^\alpha(\rho)] + (N-\rho)\frac{d}{d\rho}[L_i^\alpha(\rho)] + i\,L_i^\alpha(\rho) =$$

$$= (-1)^i\cdot[N - i(i+\alpha)]\cdot\frac{\rho^{i-1}}{(i-1)!} + \sum_{j=0}^{i-2}(-1)^j\left[A\binom{i+\alpha+1}{i-j-1} - (A+N-\rho)\binom{i+\alpha}{i-j-1} + i\binom{i+\alpha}{i-j}\right]\cdot\frac{\rho^j}{j!}$$

$$\binom{i+\alpha}{i-j} = \frac{(i+\alpha)!}{(i-j)!(\alpha+j)!} \quad ; \quad \binom{i+\alpha}{i-j-1} = \frac{(i+\alpha)!}{(i-1-j)!(\alpha+1+j)!}$$

$$\left[-(A+N-\rho)\binom{i+\alpha}{i-j-1} + i\binom{i+\alpha}{i-j}\right] = i\cdot\frac{(i+\alpha)!}{(i-j)!(\alpha+j)!} - (A+N-\rho)\frac{(i+\alpha)!}{(i-1-j)!(\alpha+1+j)!} =$$

$$= i\cdot\frac{(i+\alpha)!}{(i-j)!(\alpha+j)!} - (A+N-\rho)\frac{(i-j)(i+\alpha)!}{(i-j)!(\alpha+1+j)!} =$$

$$= \frac{(i+\alpha)!}{(i-j)!(\alpha+j)!}\left[i - \frac{(i-j)(A+N-\rho)}{\alpha+1+j}\right] = \binom{i+\alpha}{i-j}\left[i - \frac{(i-j)(A+N-\rho)}{\alpha+1+j}\right]$$

$$A\frac{d^2}{d\rho^2}[L_i^\alpha(\rho)] + (N-\rho)\frac{d}{d\rho}[L_i^\alpha(\rho)] + i\,L_i^\alpha(\rho) =$$

$$= (-1)^i\cdot[N - i(i+\alpha)]\cdot\frac{\rho^{i-1}}{(i-1)!} + \sum_{j=0}^{i-2}(-1)^j\left[A\binom{i+\alpha+1}{i-j-1} + \binom{i+\alpha}{i-j}\left[i - \frac{(i-j)(A+N-\rho)}{\alpha+1+j}\right]\right]\cdot\frac{\rho^j}{j!}$$

$$= (-1)^{i}\cdot[N- i(i+\alpha)]\cdot \frac{\rho^{i-1}}{(i-1)!} +(-1)^{i-2}A \binom{i+\alpha+1}{i-i+2-1}\cdot\frac{\rho^{i-2}}{(i-2)!} + \sum_{j=0}^{i-3} (-1)^{j}A \binom{i+\alpha+1}{i-j-1}\cdot\frac{\rho^{j}}{j!} +$$

$$+ \sum_{j=0}^{i-2} (-1)^{j}\binom{i+\alpha}{i-j}\left[i-\frac{(i-j)(A+N-\rho)}{\alpha+1+j}\right]\cdot\frac{\rho^{j}}{j!} \quad ;$$

I primi due addendi hanno somma nulla se:

$$A \binom{i+\alpha+1}{i-i+2-1}\cdot\frac{\rho^{i-2}}{(i-2)!} =A\cdot(i-1)(i+\alpha+1)\cdot\frac{\rho^{i-2}}{(i-1)!} = - [N- i(i+\alpha)]\cdot \frac{\rho^{i-1}}{(i-1)!}$$

$$A\cdot(i-1)(i+\alpha+1) = - \rho[N- i(i+\alpha)] \qquad ; \qquad A[i(i+\alpha)-\alpha-1] = \rho[i(i+\alpha)-N]$$

$$A = \rho \quad ; \quad N = \alpha+1$$

$$\rho \frac{d^2}{d\rho^2}[L_i^{\alpha}(\rho)] + (\alpha+1-\rho)\frac{d}{d\rho}[L_i^{\alpha}(\rho)] + i\, L_i^{\alpha}(\rho) =$$

$$= \sum_{j=0}^{i-3} (-1)^{j} \binom{i+\alpha+1}{i-j-1}\frac{\rho^{j+1}}{j!} + \sum_{j=0}^{i-2} (-1)^{j}\binom{i+\alpha}{i-j}\left[i-\frac{(i-j)(\alpha+1)}{\alpha+1+j}\right]\cdot\frac{\rho^{j}}{j!} =$$

$$= \sum_{j=0}^{i-3} -(-1)^{j+1} (j+1)\binom{i+\alpha+1}{i-(j+1)}\cdot\frac{\rho^{j+1}}{(j+1)!} + \sum_{j=0}^{i-2} (-1)^{j}\binom{i+\alpha}{i-j}\left[i-\frac{(i-j)(\alpha+1)}{\alpha+1+j}\right]\cdot\frac{\rho^{j}}{j!} =$$

$$= \sum_{j+1=1}^{i-2} -(-1)^{j+1} (j+1)\binom{i+\alpha+1}{i-(j+1)}\cdot\frac{\rho^{j+1}}{(j+1)!} + \sum_{j=0}^{i-2} (-1)^{j}\binom{i+\alpha}{i-j}\left[i-\frac{(i-j)(\alpha+1)}{\alpha+1+j}\right]\cdot\frac{\rho^{j}}{j!} =$$

$$= \sum_{j=1}^{i-2} -(-1)^{j}\, j\binom{i+\alpha+1}{i-j}\frac{\rho^{j}}{j!} + \sum_{j=1}^{i-2} (-1)^{j}\binom{i+\alpha}{i-j}\left[i-\frac{(i-j)(\alpha+1)}{\alpha+1+j}\right]\cdot\frac{\rho^{j}}{j!} + \binom{i+\alpha}{i}\left[i-\frac{i(\alpha+1)}{\alpha+1}\right] =$$

$$= \sum_{j=1}^{i-2} (-1)^{j}\cdot\left\{\binom{i+\alpha}{i-j}\left[i-\frac{(i-j)(\alpha+1)}{\alpha+1+j}\right] - j\binom{i+\alpha+1}{i-j}\right\}\cdot\frac{\rho^{j}}{j!} \;;$$

Calcolo del fattore tra parentesi graffe

$$\left\{\binom{i+\alpha}{i-j}\cdot\left[i-\frac{(i-j)(\alpha+1)}{\alpha+1+j}\right] - j\binom{i+\alpha+1}{i-j}\right\} = \frac{(i+\alpha)!}{(i-j)!(\alpha+j)!}\cdot\left[i-\frac{(i-j)(\alpha+1)}{\alpha+1+j}\right] - j\,\frac{(i+\alpha+1)!}{(i-j)!(\alpha+j+1)!} =$$

$$= \frac{(i+\alpha)!}{(i-j)!(\alpha+j)!}\cdot\left[i-\frac{(i-j)(\alpha+1)}{\alpha+1+j} - j\cdot\frac{i+\alpha+1}{\alpha+1+j}\right] = \frac{(i+\alpha)!}{(i-j)!(\alpha+j)!}\cdot\frac{i(\alpha+1+j)-(i-j)(\alpha+1)-j(i+\alpha+1)}{\alpha+1+j} = 0$$

Si giunge così all' equazione differenziale:

$$\rho\,\frac{d^2}{d\rho^2}[L_i^\alpha(\rho)] + (\alpha+1-\rho)\frac{d}{d\rho}[L_i^\alpha(\rho)] + i\,L_i^\alpha(\rho) = 0$$

La forma autoaggiunta di quest'equazione si ottiene considerando la derivata

$$\frac{d}{d\rho}\left\{e^{-\rho}\cdot\rho^{\alpha+1}\frac{d}{d\rho}[L_i^\alpha(\rho)]\right\} = e^{-\rho}\cdot\rho^{\alpha}\left\{(\alpha+1-\rho)\cdot\frac{d}{d\rho}[L_i^\alpha(\rho)] + \rho\,\frac{d^2}{d\rho^2}[L_i^\alpha(\rho)]\right\} = -\,e^{-\rho}\cdot\rho^{\alpha}\;i\,L_i^\alpha(\rho)$$

$$\frac{d}{d\rho}\left\{e^{-\rho}\cdot\rho^{\alpha+1}\frac{d}{d\rho}[L_i^\alpha(\rho)]\right\} + i\cdot e^{-\rho}\cdot\rho^{\alpha}\;L_i^\alpha(\rho) = 0$$

Espressione integrale della formula di Rodriguez

$$L_j^\alpha(\rho) = \frac{1}{j!}\cdot e^{\rho}\cdot\rho^{-\alpha}\cdot\frac{j!}{2\pi i}\oint_c\frac{e^{-r}\cdot r^{j+\alpha}}{(r-\rho)^{j+1}}\,dr = e^{\rho}\cdot\rho^{-\alpha}\cdot\frac{1}{2\pi i}\oint_c\frac{e^{-r}\cdot r^{j+\alpha}}{(r-\rho)^{j+1}}\,dr = f(\rho)\cdot\frac{1}{2\pi i}\oint_c\frac{e^{-r}\cdot r^{j+\alpha}}{(r-\rho)^{j+1}}\,dr$$

$$L_j^{\alpha,\,p}(\rho) = \frac{d^p}{d\rho^p}[L_j^\alpha(\rho)]$$

$$\int_0^\infty\left[L_j^\alpha(\rho)\right]^2 e^{-\rho}\cdot\rho^{\alpha}\,d\rho = \frac{\Gamma(\alpha+j+1)}{j!}$$

$$\int_0^\infty\left[\cdot L_{j-p}^{\alpha+p}(\rho)\right]^2 e^{-\rho}\cdot\rho^{\alpha+p}\,d\rho = \frac{\Gamma(\alpha+j+1)}{(j-p)!}$$

$$\frac{(j-p)!}{\Gamma(\alpha+j+1)}\cdot\int_0^\infty\left[L_{j-p}^{\alpha+p}(\rho)\right]^2 e^{-\rho}\cdot\rho^{\alpha+p}\,d\rho = 1$$

Funzione generatrice $\Phi(\rho,s) = \sum_{j=0}^{\infty} j!\cdot L_j^\alpha(\rho)\,\frac{s^j}{j!} = \sum_{j=0}^{\infty} L_j^\alpha(\rho)\;s^j =$

$$= \sum_{j=0}^{\infty} \left[\frac{1}{j!} \cdot e^{\rho} \cdot \rho^{-\alpha} \cdot \frac{j!}{2\pi i} \oint_c \frac{e^{-r} \cdot r^{j+\alpha}}{(r-\rho)^{j+1}}\, dr \right] \cdot s^j =$$

$$= \frac{e^{\rho}\, \rho^{-\alpha}}{2\pi i} \oint_c \sum_{j=0}^{\infty} \frac{e^{-r} \cdot r^{j+\alpha}}{(r-\rho)^{j+1}}\, dr\ \ s^j = \frac{e^{\rho} \cdot \rho^{-\alpha}}{2\pi i} \oint_c \frac{e^{-r} \cdot r^{\alpha}}{r-\rho} \sum_{j=0}^{\infty} \frac{(rs)^j}{(r-\rho)^j}\, dr$$

$$\sum_{j=0}^{\infty} \frac{(rs)^j}{(r-\rho)^j} = \frac{1}{1-\frac{rs}{r-\rho}} = \frac{r-\rho}{r(1-s)-\rho}$$

$$\Phi(\rho,s) = \frac{e^{\rho} \cdot \rho^{-\alpha}}{2\pi i} \oint_c \frac{e^{-r} \cdot r^{\alpha}}{r-\rho} \cdot \frac{r-\rho}{r(1-s)-\rho}\, dr = \frac{e^{\rho} \cdot \rho^{-\alpha}}{2\pi i} \oint_c \frac{e^{-r} \cdot r^{\alpha}}{r(1-s)-\rho}\, dr =$$

$$= e^{\rho} \cdot \rho^{-\alpha} \cdot \frac{1}{1-s} \cdot \frac{1}{2\pi i} \oint_c \frac{e^{-r} \cdot r^{\alpha}}{r - \frac{\rho}{1-s}}\, dr$$

polo per $r = \frac{\rho}{1-s}$; c cerchio intorno al polo

$$\Phi(\rho,s) = e^{\rho} \cdot \rho^{-\alpha-1} \cdot \frac{\rho}{1-s} \cdot e^{-\frac{\rho}{1-s}} \cdot \left(\frac{\rho}{1-s}\right)^{\alpha} = e^{\rho\left(1-\frac{1}{1-s}\right)} \cdot \left(\frac{\rho}{1-s}\right)^{\alpha+1} \cdot \rho^{-\alpha-1} = \frac{e^{-\frac{\rho s}{1-s}}}{(1-s)^{\alpha+1}}$$

$$\Phi(\rho,s) = \sum_{j=0}^{\infty} j! \cdot L_j^{\alpha}(\rho)\, \frac{s^j}{j!} = \sum_{j=0}^{\infty} L_j^{\alpha}(\rho)\, s^j = \frac{e^{-\frac{\rho s}{1-s}}}{(1-s)^{\alpha+1}}$$

$$\frac{d}{d\rho} \Phi(\rho,s) = \sum_{j=0}^{\infty} \frac{d}{d\rho} L_j^{\alpha}(\rho)\, s^j = -\frac{s}{1-s} \cdot \frac{e^{-\frac{\rho s}{1-s}}}{(1-s)^{\alpha+1}}$$

$$\frac{d^p}{d\rho^p}\Phi(\rho,s) = \sum_{j=0}^{\infty} \frac{d^p}{d\rho^p} L_j^{\alpha}(\rho)\ s^j = \left(-\frac{s}{1-s}\right)^p \cdot \frac{e^{-\frac{\rho s}{1-s}}}{(1-s)^{\alpha+1}} = (-s)^p \cdot \frac{e^{-\frac{\rho s}{1-s}}}{(1-s)^{\alpha+p+1}}$$

$$\frac{d^p}{d\rho^p}[L_j^{\alpha}(\rho)] = (-1)^p \cdot L_{j-p}^{\alpha+p}(\rho)$$

$$\frac{d^p}{d\rho^p}\Phi(\rho,s) = \sum_{j=0}^{\infty} (-1)^p \cdot L_{j-p}^{\alpha+p}(\rho)\ s^j = (-s)^p \cdot \frac{e^{-\frac{\rho s}{1-s}}}{(1-s)^{\alpha+p+1}} = \sum_{j=0}^{\infty} (-1)^p \cdot L_j^{\alpha+p}(\rho)\ s^{j+p}$$

$$\sum_{j=0}^{\infty} (-1)^p \cdot L_{j-p}^{\alpha+p}(\rho)\ s^j = \sum_{j=0}^{p} (-1)^p \cdot L_{j-p}^{\alpha+p}(\rho)\ s^j + \sum_{j=p}^{\infty} (-1)^p \cdot L_{j-p}^{\alpha+p}(\rho)\ s^j$$

$$\sum_{j=0}^{\infty} (-1)^p \cdot L_{j-p}^{\alpha+p}(\rho)\ s^j = \sum_{j=0}^{p} (-1)^p \cdot L_{j-p}^{\alpha+p}(\rho)\ s^j + \sum_{j=0}^{\infty} (-1)^p \cdot L_j^{\alpha+,p}(\rho)\ s^{j+p}$$

$$\sum_{j=0}^{p} (-1)^p \cdot L_{j-p}^{\alpha+p}(\rho)\ s^j = 0$$

$$L_j^{\alpha}(\rho) = \sum_{q=0}^{j} (-1)^q \cdot \frac{(\alpha+j)!}{q!\cdot(j-q)!(\alpha+q)!} \cdot \rho^q = (\alpha+j)! \sum_{q=0}^{j} (-1)^q \cdot \frac{1}{q!\cdot(j-q)!(\alpha+q)!} \cdot \rho^q$$

$$L_{j-p}^{\alpha+p}(\rho) = \sum_{q=0}^{j-p} (-1)^q \cdot \frac{(\alpha+p+j-p)!}{q!\cdot(j-p-q)!(\alpha+p+q)!} \cdot \rho^q$$

$$L_{j-p}^{\alpha+p}(\rho) = \sum_{q=0}^{j-p} (-1)^q \cdot \frac{(\alpha+j)!}{q!\cdot(j-p-q)!(\alpha+p+q)!} \cdot \rho^q = (\alpha+j)! \sum_{q=0}^{j-p} \frac{(-1)^q \cdot \rho^q}{q!\cdot(j-p-q)!(\alpha+p+q)!}$$

$$L_j^{\alpha}(\rho) = e^{\rho} \cdot \rho^{-\alpha} \cdot \frac{1}{2\pi i} \oint_c \frac{e^{-r} \cdot r^{j+\alpha}}{(r-\rho)^{j+1}}\, dr \qquad c = \rho + \varepsilon \cdot e^{\phi}$$

Funzione generatrice

$$\Phi(\rho,s) = \sum_{j=0}^{\infty} j!\cdot L_j^{\alpha}(\rho)\ \frac{s^j}{j!}$$

$$\Phi(\rho,s) = \sum_{j=0}^{\infty} L_j^{\alpha}(\rho)\ s^j \qquad \Phi^{(p)}(\rho,s) = \sum_{j=0}^{\infty} \frac{d^p}{d\rho^p}\left[L_j^{\alpha}(\rho)\right]\ s^j$$

$$\Phi(\rho,s) = \sum_{j=0}^{\infty} \left[\frac{1}{j!}\cdot e^{\rho}\cdot \rho^{-\alpha}\cdot\frac{j!}{2\pi i}\oint_c \frac{e^{-r}\cdot r^{j+\alpha}}{(r-\rho)^{j+1}}\,dr\right]\cdot s^j$$

$$\Phi^{(p)}(\rho,s) = \sum_{j=0}^{\infty} [(e^{\rho}\cdot \rho^{-\alpha}\cdot\frac{p!}{2\pi i}\oint_c \frac{e^{-r}\cdot r^{j+\alpha}}{(r-\rho)^{j+1}}\,dr)\quad s^j$$

$$\Phi(\rho,s) = \frac{e^{\rho}\cdot \rho^{-\alpha}}{2\pi i}\oint_c \sum_{j=0}^{\infty} \frac{e^{-r}\cdot r^{j+\alpha}}{(r-\rho)^{j+1}}\,dr\ \ s^j$$

$$\Phi(\rho,s) = \frac{e^{\rho}\cdot \rho^{-\alpha}}{2\pi i}\oint_c \frac{e^{-r}\cdot r^{\alpha}}{r-\rho}\sum_{j=0}^{\infty} \frac{(rs)^j}{(r-\rho)^j}\,dr$$

$$\sum_{j=0}^{\infty} \frac{(rs)^j}{(r-\rho)^j} = \frac{1}{1-\frac{rs}{r-\rho}} = \frac{r-\rho}{r(1-s)-\rho}$$

$$\Phi(\rho,s) = \frac{e^{\rho}\cdot \rho^{-\alpha}}{2\pi i}\int_c \frac{e^{-r}\cdot r^{\alpha}}{r-\rho}\cdot\frac{r-\rho}{r(1-s)-\rho}\,dr$$

$$\Phi(\rho,s) = \frac{e^{\rho}\cdot \rho^{-\alpha}}{2\pi i}\int_c \frac{e^{-r}\cdot r^{\alpha}}{r(1-s)-\rho}\,dr$$

$$\Phi(\rho,s) = e^{\rho}\cdot \rho^{-\alpha}\cdot\frac{1}{1-s}\cdot\frac{1}{2\pi i}\oint_c \frac{e^{-r}\cdot r^{\alpha}}{r-\frac{\rho}{1-s}}\,dr \;;\qquad \Phi^{(p)}(\rho,s) = e^{\rho}\cdot \rho^{-\alpha}\cdot\frac{1}{1-s}\cdot\frac{p!}{2\pi i}\oint_c \frac{e^{-r}\cdot r^{\alpha}}{\left(r-\frac{\rho}{1-s}\right)^p}\,dr$$

polo per r = $\frac{\rho}{1-s}$;

$$\Phi^{(p)}(\rho,s) = e^{\rho}\cdot\rho^{-\alpha-1}\cdot\left(\frac{\rho}{1-s}\right)^{\alpha+1}\cdot\left(1-\frac{1}{1-s}\right)^{p}\cdot e^{-\frac{\rho}{1-s}}$$

c cerchio intorno al polo

$$\Phi^{(p)}(\rho,s) = e^{\rho}\cdot\rho^{-\alpha-p-1}\cdot\left(\frac{\rho}{1-s}\right)^{\alpha+1}\cdot\left(\rho-\frac{\rho}{1-s}\right)^{p}\cdot e^{-\frac{\rho}{1-s}}$$

$$\Phi^{(p)}(\rho,s) = e^{\rho}\cdot\rho^{-\alpha-p-1}\cdot\left(\frac{\rho}{1-s}\right)^{\alpha+1}\cdot\left(\frac{s\rho}{1-s}\right)^{p}\cdot e^{-\frac{\rho}{1-s}}.$$

$$\Phi(\rho,s) = e^{\rho}\cdot\rho^{-\alpha-1}\cdot\frac{\rho}{1-s}\cdot e^{-\frac{\rho}{1-s}}\cdot\left(\frac{\rho}{1-s}\right)^{\alpha} \;;\qquad \Phi^{(p)}(\rho,s) = e^{\rho}\cdot e^{-\frac{\rho}{1-s}}\cdot(-s)^{p}\cdot\left(\frac{\rho}{1-s}\right)^{\alpha+p+1}\cdot\rho^{-\alpha-p-1}$$

$$\Phi(\rho,s) = e^{\rho\left(1-\frac{1}{1-s}\right)}\cdot\left(\frac{\rho}{1-s}\right)^{\alpha+1}\cdot\rho^{-\alpha-1};\qquad \Phi^{(p)}(\rho,s) = (-s)^{p}\cdot e^{\rho\left(1-\frac{1}{1-s}\right)}\cdot\left(\frac{\rho}{1-s}\right)^{\alpha+p+1}\cdot\rho^{-\alpha-p-1}$$

$$\Phi(\rho,s) = \frac{e^{-\frac{\rho s}{1-s}}}{(1-s)^{\alpha+1}}\qquad \Phi^{(p)}(\rho,s) = (-s)^{p}\cdot\frac{e^{-\frac{\rho s}{1-s}}}{(1-s)^{\alpha+p+1}}$$

$$\Phi(\rho,s) = \sum_{j=0}^{\infty} j!\cdot L_j^{\alpha}(\rho)\,\frac{s^j}{j!} = \sum_{j=0}^{\infty} L_j^{\alpha}(\rho)\, s^j = \frac{e^{-\frac{\rho s}{1-s}}}{(1-s)^{\alpha+1}}$$

$$L_j^{\alpha+p}(\rho) = \frac{1}{j!}\cdot\sum_{q=0}^{j}(-1)^q\cdot\binom{j}{q}\frac{(\alpha+p+j)!}{(\alpha+p+q)!}\cdot\rho^q$$

$$\frac{d^p}{d\rho^p}\Phi(\rho,s) = \Phi^{(p)}(\rho,s) = (-1)^p\cdot\sum_{j=0}^{\infty} L_j^{\alpha+p}(\rho)\, s^j = (-s)^p\cdot\frac{e^{-\frac{\rho s}{1-s}}}{(1-s)^{\alpha+p+1}} =$$

$$= (-1)^p\cdot\sum_{j=0}^{\infty}\frac{1}{j!}\cdot\sum_{q=0}^{j}(-1)^q\cdot\binom{j}{q}\frac{(\alpha+p+j)!}{(\alpha+p+q)!}\cdot\rho^q\; s^{j+p} =$$

$$= (-s)^p\cdot\sum_{q=0}^{j}\sum_{j=0}^{\infty}(-1)^q\cdot\frac{1}{j!}\cdot\binom{j}{q}\frac{(\alpha+p+j)!}{(\alpha+p+q)!}\cdot\rho^q\; s^j$$

$$=(-1)^p s^p\sum_{q=0}^{j}\frac{\rho^q(-1)^q}{(\alpha+p+q)!}\sum_{j=0}^{\infty}\frac{1}{j!}\cdot\binom{j}{q}(\alpha+p+j)!\; s^j$$

$$=(-1)^p s^p \sum_{q=0}^{j} \frac{\rho^q(-1)^q}{(\alpha+p+q)!} \sum_{j=0}^{\infty} \frac{(\alpha+p+j)!}{q!\cdot(j-q)!} s^j$$

$$L_j^{\alpha}(\rho) = \sum_{q=0}^{j} (-1)^q \cdot \frac{(\alpha+j)!}{q!\cdot(j-q)!(\alpha+q)!} \cdot \rho^q$$

$$\int_0^{\infty} \left[L_i^{\alpha}(\rho)\right]^2 e^{-\rho} \cdot \rho^{\alpha}\, d\rho = \frac{\Gamma(\alpha+i+1)}{i!}$$

$$-(i+1)L_{i+1}^{\alpha}(\rho) + (\alpha+2i+1-\rho)\, L_i^{\alpha}(\rho) - (\alpha+i)\, L_{i-1}^{\alpha}(\rho) = 0$$

$$L_{j-p}^{\alpha+p}(\rho) = \sum_{q=0}^{j-p} (-1)^q \cdot \frac{(\alpha+j)!}{q!\cdot(j-p-q)!(\alpha+p+q)!} \cdot \rho^q = (\alpha+j)! \sum_{q=0}^{j-p} \frac{(-1)^q \cdot \rho^q}{q!\cdot(j-p-q)!(\alpha+p+q)!}$$

$$\frac{d^p}{d\rho^p}\Phi(\rho,s) = \sum_{j=0}^{\infty} (-1)^p \cdot L_{j-p}^{\alpha+p}(\rho)\, s^j = (-s)^p \cdot \frac{e^{-\frac{\rho s}{1-s}}}{(1-s)^{\alpha+p+1}}$$

$$= (-1)^p \cdot \sum_{j=0}^{\infty} (\alpha+j)!\, s^p \sum_{q=0}^{j-p} \frac{(-1)^q \cdot \rho^q}{q!\cdot(j-p-q)!(\alpha+p+q)!} s^{j-p} = \qquad j \geq p$$

$j-p = k$

$$= (-s)^p \cdot \sum_{k+p=0}^{\infty} (\alpha + k+p)! \sum_{q=0}^{k} \frac{(-1)^q \cdot \rho^q}{q! \cdot (k-q)!(\alpha+p+q)!} s^k = \qquad ; k \geq 0$$

$$= (-s)^p \cdot \sum_{k+p=0}^{\infty} \sum_{q=0}^{k} \frac{(-1)^q \cdot \rho^q (\alpha+k+p)!}{q! \cdot (k-q)!(\alpha+p+q)!} s^k =$$

$$= (-s)^p \cdot \sum_{q=0}^{j-p} \frac{(-1)^q \cdot \rho^q}{q! \cdot (\alpha+p+q)!} \sum_{j=0}^{\infty} \frac{(\alpha + j)!}{(j-p-q)!} s^{j-p} = \qquad j \geq p+q \qquad j-p-q = k$$

$$= (-1)^p \cdot \sum_{q=0}^{j-p} \frac{(-1)^q \cdot \rho^q}{q! \cdot (\alpha+p+q)!} \sum_{j=p+q}^{\infty} \frac{(\alpha + j)!}{(j-p-q)!} s^j =$$

$$\sum_{j=0}^{\infty} \frac{(rs)^j}{(r-\rho)^j} = \frac{1}{1-\frac{rs}{r-\rho}} = \frac{r-\rho}{r(1-s)-\rho}$$

$$\Phi(\rho,s) = \sum_{j=0}^{\infty} j! \cdot L_j^{\alpha}(\rho) \frac{s^j}{j!} = \sum_{j=0}^{\infty} L_j^{\alpha}(\rho) s^j = \frac{e^{-\frac{\rho s}{1-s}}}{(1-s)^{\alpha+1}}$$

$$=(-1)^p s^p \sum_{q=0}^{j} \frac{\rho^q(-1)^q}{(\alpha+p+q)!} \sum_{j=0}^{\infty} \frac{1}{j!} \cdot \binom{j}{q} (\alpha+p+j)!\, s^j$$

$$=(-1)^p s^p \sum_{q=0}^{j} \frac{\rho^q(-1)^q}{(\alpha+p+q)!} \sum_{j=0}^{\infty} \frac{(\alpha+p+j)!}{q!\cdot(j-q)!}\, s^j$$

Applicando le formule di derivazione dei polinomi ortogonali classici, la derivata prima è espressa da:

$$L_i^{\alpha,1}(\rho) = \frac{d}{d\rho}[L_i^{\alpha}(\rho)] = -\,L_{i-1}^{\alpha+1}(\rho)$$

e per applicazioni ricorsive:

$$L_i^{\alpha,\,p}(\rho) = (-1)^p \cdot L_{i-p}^{\alpha+p}(\rho)$$

Schiff

$$L_q^p(\rho) = \sum_{k=0}^{q-p} (-1)^{k+1} \frac{(q!)^2 \rho^k}{(q-p-k)!\cdot(p-k)!\cdot k!}$$

$$\frac{L_q^p(\rho)}{q!} = \sum_{k=0}^{q-p} (-1)^{k+1} \frac{q!\,\rho^k}{(q-p-k)!\cdot(p-k)!\cdot k!}$$

$$\int_0^{\infty} \left[\cdot L_{j-p}^{\alpha+p}(\rho)\right]^2 e^{-\rho} \cdot \rho^{\alpha+p}\, d\rho = \frac{\Gamma(\alpha+j+1)}{(j-p)!}$$

$$\frac{(j-p)!}{\Gamma(\alpha+j+1)} \cdot \int_0^{\infty} \left[L_{j-p}^{\alpha+p}(\rho)\right]^2 e^{-\rho} \cdot \rho^{\alpha+p}\, d\rho = 1$$

$$\sum_{q=p}^{\infty} \frac{L_q^p(\rho)}{q!} s^q = \sum_{q=p}^{\infty} \sum_{k=0}^{q-p} (-1)^{k+1} \frac{\rho^k q! s^q}{(q-p-k)!\cdot(p-k)!\cdot k!}$$

$$\sum_{q=p}^{\infty} \frac{L_q^p(\rho)}{q!} s^q = \sum_{k=0}^{q-p} \sum_{q=p}^{\infty} (-1)^{k+1} \frac{\rho^k q! s^q}{(q-p-k)!\cdot(p-k)!\cdot k!}$$

$$L_2^{\alpha}(\rho) = \frac{1}{2} \cdot[\rho^2 - 2(\alpha+2)\cdot\rho + (\alpha+2)(\alpha+1)]$$

$$L_2^{\alpha-1}(\rho) = \frac{1}{2} \cdot[\rho^2 - 2(\alpha+1)\cdot\rho + (\alpha+1)\alpha]$$

$$L_3^{\alpha}(\rho) = \frac{1}{6}[(-1)^3 \rho^3 + 3(-1)^2(\alpha+3)\cdot\rho^2 + 3(-1)^1(\alpha+3)(\alpha+2)\,\rho + (-1)^0(\alpha+3)(\alpha+2)(\alpha+1)]$$

$$L_2^{\alpha}(\rho) = \frac{1}{6} \cdot[3\rho^2 - 6(\alpha+2)\cdot\rho + 3(\alpha+2)(\alpha+1)]$$

$$L_3^{\alpha}(\rho) - L_2^{\alpha}(\rho) = \frac{1}{6} \cdot\{-\rho^3 + 3(\alpha+2)\cdot\rho^2 - 3[(\alpha+3)(\alpha+2) - 2(\alpha+2)]\,\rho + (\alpha+3-3)(\alpha+2)(\alpha+1)\} =$$

$$= \frac{1}{6} \cdot[-\rho^3 + 3(\alpha+2)\cdot\rho^2 - 3(\alpha+1)(\alpha+2)\,\rho + \alpha(\alpha+2)(\alpha+1)] = L_3^{\alpha-1}(\rho)$$

$$\frac{1}{6}[-\rho^3 + 3(\alpha+3)\cdot\rho^2 - 3(\alpha+3)(\alpha+2)\,\rho + (\alpha+3)(\alpha+2)(\alpha+1)] = L_3^{\alpha}(\rho)$$

$$L_3^{\alpha}(\rho) - L_2^{\alpha}(\rho) = L_3^{\alpha-1}(\rho) \quad ; \quad L_j^{\alpha}(\rho) - L_{j-1}^{\alpha}(\rho) = L_j^{\alpha-1}(\rho)$$

$$\sum_{q=0}^{j} L_q^{\alpha}(\rho) = L_j^{\alpha+1}(\rho)$$

$$H_0(x) = 1 = (x^2)^0$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2 = 4(x^2)^1 - 2$$

$$H_3(x) = 4(2x^3 - 3x) = 4x[2(x^2)^1 - 3]$$

$$H_4(x) = 4(4x^4 - 12x^2 + 3) = 4[4(x^2)^2 - 12(x^2)^1 + 3]$$

$$H_5(x) = 8(4x^5 - 20x^3 + 15x) = 8x[4(x^2)^2 - 20(x^2)^1 + 15]$$

$$L_0^{-\frac{1}{2}}(\rho) = 1$$

$$L_1^{-\frac{1}{2}}(\rho) = \frac{1}{2} - \rho$$

$$L_2^{-\frac{1}{2}}(\rho) = \frac{1}{2} \cdot [\frac{3}{4} - 3\rho + \rho^2]$$

$$L_3^{-\frac{1}{2}}(\rho) = \frac{1}{6} \cdot [\frac{15}{8} - \frac{45}{4}\rho + \frac{15}{2}\rho^2 - \rho^3]$$

$$L_4^{-\frac{1}{2}}(\rho) = \frac{1}{24} \cdot [\frac{105}{16} - \frac{105}{2}\rho + \frac{105}{2}\rho^2 - 14\rho^3 + \rho^4]$$

$$L_5^{-\frac{1}{2}}(\rho) = \frac{1}{120} \cdot [\frac{945}{32} - \frac{4725}{16}\rho + \frac{3150}{8}\rho^2 - \frac{630}{4}\rho^3 + \frac{45}{2}\rho^4 - \rho^5]$$

$$L_0^{-\frac{1}{2}}(\rho) = 1$$

$$2 \cdot L_1^{-\frac{1}{2}}(\rho) = 1 - 2\rho$$

$$8 \cdot L_2^{-\frac{1}{2}}(\rho) = 3 - 12\rho + 4\rho^2$$

$$48 \cdot L_3^{-\frac{1}{2}}(\rho) = 15 - 90\rho + 60\rho^2 - 8\rho^3$$

$$384 \cdot L_4^{-\frac{1}{2}}(\rho) = 105 - 840\rho + 840\rho^2 - 224\rho^3 + 16\rho^4$$

$$3840 \cdot L_5^{-\frac{1}{2}}(\rho) = 945 - 9450\,\rho + 12600\,\rho^2 - 5040\rho^3 + 720\rho^4 - 32\rho^5$$

$$L_0^{\frac{1}{2}}(\rho) = 1$$

$$L_1^{\frac{1}{2}}(\rho) = \frac{3}{2} - \rho$$

$$L_2^{\frac{1}{2}}(\rho) = \frac{1}{2} \cdot \left[\frac{15}{4} - 5\rho + \rho^2\right]$$

$$L_3^{\frac{1}{2}}(\rho) = \frac{1}{6} \cdot \left[\frac{105}{8} - \frac{105}{4}\rho + \frac{21}{2}\rho^2 - \rho^3\right]$$

$$L_4^{\frac{1}{2}}(\rho) = \frac{1}{24} \cdot \left[\frac{945}{16} - \frac{315}{2}\rho + \frac{189}{2}\rho^2 - 18\rho^3 + \rho^4\right]$$

$$L_5^{\frac{1}{2}}(\rho) = \frac{1}{120} \cdot \left[\frac{10395}{32} - \frac{17325}{16}\rho + \frac{6930}{8}\rho^2 - \frac{990}{4}\rho^3 + \frac{55}{2}\rho^4 - \rho^5\right]$$

$$x^2 = \rho$$

$$H_0(x) = 1 = (x^2)^0 = 1$$

$$H_2(x) = 4x^2 - 2 = 4(x^2)^1 - 2 = 4\rho - 2$$

$$H_4(x) = 4(4x^4 - 12x^2 + 3) = 4[4\rho^2 - 12\rho + 3]$$

$$L_0^{-\frac{1}{2}}(\rho) = 1$$

$$2L_1^{-\frac{1}{2}}(\rho) = 1 - 2\rho$$

$$8L_2^{-\frac{1}{2}}(\rho) = 3 - 12\rho + 4\rho^2$$

$$\frac{1}{x} \cdot H_1(x) = 2$$

$$\frac{1}{x} \cdot H_3(x) = 4 \cdot \frac{1}{x} \cdot (2x^3 - 3x) = 4 \cdot [2(x^2)^1 - 3] = 4[2\rho - 3]$$

$$\frac{1}{x} \cdot H_5(x) = 8 \cdot \frac{1}{x} \cdot (4x^5 - 20x^3 + 15x) = 8 \cdot [4(x^2)^2 - 20(x^2)^1 + 15] = 8[4\rho^2 - 20\rho + 15]$$

$$2\cdot L_1^{\frac{1}{2}}(\rho) = 3 - 2\rho$$

$$8\cdot L_2^{\frac{1}{2}}(\rho) = 15 - 20\rho + 4\rho^2$$

$$H_0(x) = L_0^{-\frac{1}{2}}(\rho)$$

$$H_2(x) = -4\cdot L_1^{-\frac{1}{2}}(\rho)$$

$$H_4(x) = 32\cdot L_2^{-\frac{1}{2}}(\rho)$$

$$\frac{1}{x}\cdot H_3(x) = -8\cdot L_1^{\frac{1}{2}}(\rho)$$

$$\frac{1}{x}\cdot H_5(x) = 64\cdot L_2^{\frac{1}{2}}(\rho)$$

$$L_i^{\alpha}(\rho) = \sum_{h=0}^{i} (-1)^h \binom{i+\alpha}{i-h} \cdot \frac{\rho^h}{h!}$$

$$L_j^{\alpha}(\rho) = \sum_{k=0}^{j} (-1)^k \binom{j+\alpha}{j-k} \cdot \frac{\rho^k}{k!}$$

$$C_{i,j}^{\alpha} \int_0^{\pi} d\theta \int_{-\pi}^{\pi} F_i(\rho,\theta,\phi,\alpha)\cdot F_j(\rho,\theta,\phi,\alpha)\, d\phi = e^{-\frac{\rho}{2}}\cdot\rho^{\frac{\alpha}{2}} L_i^{\alpha}(\rho)\cdot e^{-\frac{\rho}{2}}\cdot\rho^{\frac{\alpha}{2}} L_j^{\alpha}(\rho)$$

$$e^{-\frac{\rho}{2}}\cdot\rho^{\frac{\alpha}{2}} L_i^{\alpha}(\rho) = e^{-\frac{\rho}{2}}\cdot \sum_{h=0}^{i} (-1)^h \binom{i+\alpha}{i-h} \cdot \frac{\rho^{h+\frac{\alpha}{2}}}{h!}$$